FOURIER SERIES AND TRANSFORMS IN GRAND LEBESGUE SPACES

As an particular case - exponential Orlicz spaces.

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Abstract. In this article we investigate the Fourier series and transforms for the functions defined on the $[-\pi, \pi]^d$ or on the R^d and belonging to the (Bilateral) Grand Lebesgue Spaces.

As a particular case we obtain some results about Fourier's transform in the so-called exponential Orlicz spaces.

We construct also several examples to show the exactness of offered estimations.

Key words and phrases: Grand and ordinary Lebesgue Spaces (GLS), Hilbert transform, Orlicz and other rearrangement invariant (r.i.) spaces, Fourier integrals and series, operators, moment and Leindler inequalities, equivalent norms, upper and lower estimations, slowly varying functions, Wavelets, Haar's Series.

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1 Introduction. Notations. Problem Statement.

For the and real valued measurable function $f = f(x)$, defined on the $X = \{x\} = T^d = [-\pi, \pi]^d$ or equally $X = [0, \ 2\pi]^d, \ d = 1, 2, \dots$ or $X = R^d$ we denote correspondingly the Fourier coefficients and transform

$$c(n) = \int_{T^d} \exp(i(n, x)) \ f(x) \ dx, \quad F[f](t) = \int_{R^d} \exp(i(t, x)) \ f(x) \ dx,$$

where as usually

$$F[f](t) \stackrel{def}{=} \lim_{M \to \infty} \int_{|x| \leq M} \exp(i(t, x)) \ f(x) \ dx,$$

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$$x = (x_1, x_2, \ldots, x_d), \ n = (n_1, n_2, \ldots, n_d), \ t = (t_1, t_2, \ldots, t_d), \ dx = \prod_{j=1}^{d} dx_j$$

if $x \in R^d$, and $dx = (2\pi)^{-d} \prod_{j=1}^{d} dx_j$ in the case $X = T^d$;

$$n_i = 0, \pm 1, \pm 2, \ldots, \ (t, x) = \sum_{j=1}^{d} t_j x_j, \ |n| = \max_j |n_j|, \ |t| = \max_j |t_j|,$$

$$s_M[f](x) = (2\pi)^{-d} \sum_{n:|n|\leq M} c(n) \exp(-i(n, x)), \ x \in T^d; \tag{1.1}$$

$$S_M[f](x) = (2\pi)^{-d} \int_{t:|t|\leq M} \exp(-i(t, x)) \ F[f](t) \ dt, \ x \in R^d. \tag{1.2}$$

Our aim in this paper is investigating of the boundedness of (linear) operators $S_M[\cdot]$, $s_M[\cdot]$ in some Grand Lebesgue Spaces norms $||\cdot||G\psi$ (see definition further):

$$\sup_{M\geq 2} \ \sup_{f:||f||G\psi=1} ||s_M[f]||G\psi_1 < \infty, \ \sup_{M\geq 2} \ \sup_{f:||f||G\psi=1} ||S_M[f]||G\psi_1 < \infty, \tag{1.3}$$

and the convergence and divergence in this norms

$$s_M[f](\cdot) \to f, \ S_M[f](\cdot) \to f \tag{1.4}$$

as $M \to \infty$.

We will prove that the so-called exponential Orlicz spaces over X are the particular cases of the Grand Lebesgue Spaces. Therefore, we can consider also that the function $f(\cdot)$ belongs to some Orlicz space $L(N) = L(N; X)$ with so - called *exponential $N - function$ $N = N(u)$*, and will investigate the properties of Fourier transform of f, for example, the boundedness of operators $S_M[\cdot]$, $s_M[\cdot]$ and the convergence and divergence (1)in *some* Orlicz norms $L(N; X)$.

Note than the case if the function $N(\cdot)$ satisfies the Δ_2 condition is known; see, for example, [36], [35]. Our results are also some generalization of [12], [25], [33], [46] etc.

The papier is organized as follows. In the next section we recall used facts about Grand Lebesgue Spaces and obtain some new properties of this spaces, especially, investigate the properties of these spaces in the case when the measure is discrete. In the third section we obtain the GLS boundedness of Hilbert's transform.

The fourth section is devoted to the weight Fourier operators boundedness in GLS spaces. The 5th section contain the main result of the offered papier: the boundedness of Fourier transforms in GLS spaces in general case, for instance, in

the exponential Orlicz spaces. In the next section we formulate and prove some auxiliary facts.

The 7th section contain the proofs of main results.

In the last section we prove the GLS boundedness of the so-called maximal Fourier operators.

In many offered estimations we show their exactness by means of construction of suitable (counter) examples.

2 Grand Lebesgue Spaces.

Now we will describe using Grand Lebesgue Spaces (GLS) and a particular case the so-called Exponential Orlicz Spaces (EOS).

1. Description of used Classical Lebesgue Spaces.

Let (X, A, μ) be some measurable space with sigma-finite non - trivial measure μ. For the measurable real valued function $f(x), \ x \in X, f : X \to R$ the symbol $|f|_p = |f|_p(X, \mu)$ will denote the usually L_p norm:

$$|f|_p = ||f|| L_p(X, \mu) = \left[\int_X |f(x)|^p \ \mu(dx) \right]^{1/p}, \ p \geq 1. \tag{2.1}$$

In the case $X = R^d$ we introduce a new measure $\nu(\cdot)$ (non - finite, in general case): for all Borel set $A \subset R^d$

$$\nu(A) = \int_A \prod_{i=1}^d x_i^{-2} \ dx = \int_A \prod_{i=1}^d x_i^{-2} \cdot \prod_{i=1}^d dx_i, \tag{2.2}$$

and will denote $|f|_p(\nu) =$

$$\left\| \left(\prod_{i=1}^d x_i \right) \cdot f \right\| L_p(X, \nu) = \left[\int_X \left| \prod_{j=1}^d x_j \right|^p \cdot |f(x)|^p \ \nu(dx) \right]^{1/p} =$$

$$\left[\int_X \left| \prod_{j=1}^d x_j \right|^{p-2} \cdot |f(x)|^p \ dx \right]^{1/p}. \tag{2.3}$$

For arbitrary multiply sequence (complex, in general case) $c(n) = c(n_1, n_2, \ldots, n_d), n_i = 0, \pm 1, \pm 2, \ldots, n \in Z^d$ we denote as usually

$$|c|_p = \left[\sum_n |c(n)|^p \right]^{1/p}, \ p \geq 1; \tag{2.4}$$

and introduce the discrete analog of $|f|_p(\nu)$ norm:

$$|c|_{p,\nu} = |c|_{p,\nu}^{(d)} = \left[\sum_n |c(n)|^p \cdot \left(\left| \prod_{j=1}^d n_j \right|^{p-2} + 1 \right) \right]^{1/p}, \ p \geq 2. \tag{2.5}$$

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2. Grand Lebesgue Spaces.

We recall in this section for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [9], [10], [11], [14], [15], [18], [24], [25], [26], [27], [28], [29], [31] etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}, A \geq 1, A < B \leq \infty$, spaces consisting on all the measurable functions $f : T \to R$ with finite norms

$$||f||G(\psi) \stackrel{def}{=} \sup_{p \in (A,B)} [||f|_p/\psi(p)]. \tag{2.6}$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A,B)} \psi(p) > 0, \ \psi(p) = \infty, \ p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{def}{=} (A, B) = \{p : \psi(p) < \infty, \} \tag{2.7}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [2], and are used, for example, in the theory of probability [39], [18], [24]; theory of Partial Differential Equations [10], [15]; functional analysis [27], [28]; theory of Fourier series [30], theory of martingales [25] etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, B)$, a function $p \to p \cdot \log \psi(p)$ is convex, and $B = \infty$, then the space $G\psi$ coincides with some *exponential* Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marzinkievitch etc.

We will use the following two important examples (more exact, the *two families of examples* of the ψ functions and correspondingly the GLS spaces.

1. We denote

$$\psi(A, B; \alpha, \beta; p) \stackrel{def}{=} (p - A)^{-\alpha} (B - p)^{-\beta}, \tag{2.8}$$

where $\alpha, \beta = \text{const} \geq 0, 1 \leq A < B < \infty; p \in (A, B)$ so that

$$\text{supp} \, \psi(A, B; \alpha, \beta; \cdot) = (A, B).$$

2. Second example:

$$\psi(1, \infty; 0, -\beta; p) \stackrel{def}{=} p^\beta, \tag{2.9}$$

but here $\beta = \text{const} > 0, \ p \in (1, \infty)$ so that

$$\text{supp} \, \psi(1, \infty; 0, -\beta; \cdot) = (1, \infty).$$

The space $G\psi(1, \infty; 0, -\beta; \cdot)$ coincides up to norm equivalence with the Orlicz space over the set D with usually Lebesgue measure and with the correspondent $N(\cdot)$ function

$$N(u) = \exp\left(u^{1/\beta}\right), \ u \geq 1; N(u) = C|u|, |u| \leq 1.$$

Recall that the domain D has finite measure; therefore the behavior of the function $N(\cdot)$ is'nt essential.

Remark 1. If we define the *degenerate* $\psi_r(p), r = \text{const} \geq 1$ function as follows:

$$\psi_r(p) = \infty, \ p \neq r; \psi_r(r) = 1$$

and agree $C/\infty = 0, C = \text{const} > 0$, then the $G\psi_r(\cdot)$ space coincides with the classical Lebesgue space L_r.

Remark 2. Let $\xi : D \to R$ be some (measurable) function from the set $L(p_1, p_2), \ 1 \leq p_1 < p_2 \leq \infty$. We can introduce the so-called *natural* choice $\psi_\xi(p)$ as as follows:

$$\psi_\xi(p) \stackrel{def}{=} |\xi|_p; \ p \in (p_1, p_2).$$

3. Discrete Grand Lebesgue Spaces.

A. GENERAL PART.

Let $c = \vec{c} = \{c(1), c(2), c(3), \ldots, c(n), \ldots\}$ be arbitrary numerical sequence, $\beta = \vec{\beta} = \{\beta(1), \beta(2), \beta(3), \ldots, \beta(n), \ldots\}$ be arbitrary non-negative non-trivial:

$$\sum_{n=1}^{\infty} \beta(n) \in (0, \infty]$$

numerical sequence, $p \in (A, B), \ 1 \leq A < B \leq \infty, \ \psi : (A, B) \to R_+, \ \psi \in \Psi(A, B)$. We define as before the so-called *weight discrete* GLS space $G_d\psi_\beta(A, B) = G_d\psi_\beta$ as a set of numerical sequences with finite norm

$$||c||G_d\psi_\beta = \sup_{p \in (A,B)} \left[\frac{|c|_{p,\beta}}{\psi(p)}\right], \tag{2.10}$$

where

$$|c|_{p,\beta} \stackrel{def}{=} \left[\sum_{n=1}^{\infty} |c(n)|^p \ \beta(n)\right]^{1/p}. \tag{2.11}$$

Evidently, the $G_d\psi_\beta$ spaces are particular cases of general GLS spaces, relative the weight measure

$$\mu_\beta(A) = \sum_{k \in A} \beta(k).$$

But this spaces are *resonant spaces* in the terminology of the book [2] only in the case when $\beta(n) = \text{const} > 0$. We can suppose in this case without loss of generality that $\beta(n) = 1$ and will write for simplicity

$$||c||G_d\psi = \sup_{p \in (A,B)} \left[\frac{|c|_p}{\psi(p)}\right],$$

where as ordinary

$$|c|_p \overset{def}{=} \left[\sum_{n=1}^{\infty} |c(n)|^p \right]^{1/p}.$$

B. NATURAL FUNCTION.

Let $c = \vec{c}$ be the numerical sequence such that for some number

$$\exists p_0 \in [1, \infty) \ |c|_{p_0} < \infty. \tag{2.12}$$

We investigate in this pilcrow the natural function $\psi_c(p)$ for the sequence $\vec{c}$:

$$\psi_c(p) = |c|_p = \left[\sum_{n=1}^{\infty} |c(n)|^p \right]^{1/p},$$

in addition to the assertions of the pilcrow 2.

Note first of all that if $q > p \geq 1$, then $|c|_q \leq |c|_p$. Therefore, if for some $p_0 \in [1, \infty) \ |c|_{p_0} < \infty$, then for all the values p, $p > p_0 \Rightarrow \psi_c(p) < \infty$.

Further,

$$\lim_{n \to \infty} c(n) = 0,$$

and

$$\lim_{p \to \infty} |c|_p = \sup_n |c(n)| = \max_n |c(n)| \overset{def}{=} |c|_\infty.$$

Thus, we proved the following assertion.

Lemma 1. Every non-trivial natural discrete function $\psi = \psi(p)$ has the following properties:

0. The domain of definition of the function $\psi(\cdot)$ is some semi-axis (p_0, ∞) or $[p_0, \infty)$, where $p_0 \geq 1$.

1. The function $\psi(\cdot)$ is monotonically non-increasing.

The proposition of the Lemma 1 is false in the case of weighted discrete GLS spaces. Let us consider the correspondent example.

Example 1. Let us consider the following weight sequence $\beta^{(s)}$:

$$\beta^{(s)}(n) = n^{-1-s},$$

and the following numerical sequence $y = \{y(n)\}$, where

$$y(n) = n^\theta.$$

Here $s, \theta = \text{const}, p_0 \overset{def}{=} s/\theta > 1$.

Note that the norm $||y||_{p,\beta^{(s)}}, \ p \geq 1$ is finite only when $p < p_0$:

$$||y||_{p,\beta^{(s)}}^p = \sum_{n=1}^{\infty} n^{-1-s+p\theta} < \infty \iff p < s/\theta = p_0.$$

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Example 2. Let us consider the following sequence: $a = a^{(L)} = \{a(n)\}, a(n) = n^{-1/L}, \; L = \text{const} \geq 1$. We have for the values $p > L$, denoting

$$\psi_{(L)}(p) = \psi_{a^{(L)}}(p) :$$

$$\psi_{(L)}^p(p) = \sum_{n=1}^{\infty} n^{-p/L}.$$

The last expression coincides with the well-known Rieman's zeta-function at the point p/L. Therefore,

$$\psi_{(L)}(p) \asymp \left[\frac{p}{p-L} \right]^{1/L}, \; p \in (L, \infty). \tag{2.13}$$

C. Tail behavior.

Let $c = \vec{c} \in G_d \psi(a, b), \; 1 \leq a < b \leq \infty$. We introduce as ordinary the tail function $T_\beta(c, u), u \in (0, \infty)$ for the sequence $\{c\}$ relative an arbitrary discrete measure μ_β :

$$T_\beta(c, u) \stackrel{def}{=} \mu_\beta(n : |c(n)| \geq u). \tag{2.14}$$

We will write for simplicity in the case $\beta(n) = \beta_0(n) = 1, n = 1, 2, \ldots$

$$T_{\beta_0}(c, u) \stackrel{def}{=} T(c, u) = \text{card}\{n : |c(n)| > u\}.$$

If the sequence $\{|c(n)|\}$ is bounded, for example, if for some $p \geq 1 \; |c|_p < \infty$,

$$\forall u > \sup_n |c(n)| \; \Rightarrow T(c, u) = 0.$$

Therefore, we must investigate *in this case* the asymptotical behavior of the tail function $T(c, \epsilon)$ only as $\epsilon \to 0 +$.

It follows from Tchebychev's inequality that

$$T_\beta(c, u) \leq \inf_{p \in (a,b)} \left[||f||_{p,\beta}^p \psi^p(p)/u^p \right], \; u > 0. \tag{2.15}$$

Conversely,

$$|c|_{p,\beta}^p = p \int_0^\infty u^{p-1} T_\beta(c, u) du, \; p \geq 1;$$

therefore

$$||c|| G_d(\psi) = \sup_{p:\psi(p)<\infty} \left[p \left[\int_0^\infty u^{p-1} T_\beta(c, u) \, du \right]^{1/p} /\psi(p) \right]. \tag{2.16}$$

In order to show the exactness of this inequalities, we consider some examples.

EXAMPLE 1. Let

$$a(k) = k, \; \beta(k) = k^{-b-1}, \; b = \text{const} > 1.$$

We find by the direct calculations:

$$|a|_{p,\beta}^p \sim (b-p)^{-1}, \ p \to b-0;$$

$$T_\beta(a,u) = \sum_{k \geq u} k^{-b-1} \sim u^{-b}/b, \ u \to \infty;$$

but it follows from the upper estimation for the tail function that as $u \to \infty$

$$T_\beta(a,u) \leq C \inf_{p \in (1,b)} \left[u^{-p}/(b-p) \right] \sim C_1 u^{-b} \ \log u.$$

More generally, if

$$\beta(k) = \beta^{(\Delta)}(k) = k^{-b-1} \ \log^\Delta(k),$$

$b = \text{const} > 1, \Delta = \text{const} \geq 0, \ a(k) = k,$ and $p \to b-0$, then

$$|a|_{p,\beta}^p = \sum_{k=1}^\infty k^{-b-1+p} \log^\Delta k \sim$$

$$\int_1^\infty x^{p-b-1} \ \log^\Delta(x) \ dx = \frac{\Gamma(\Delta+1)}{(b-p)^\gamma}, \ \gamma \stackrel{def}{=} \Delta + 1; \tag{2.17}$$

$$T_{\beta^{(\Delta)}}(a,u) = \sum_{k \geq u} k^{-b-1} \log^\Delta(k) \sim$$

$$b^{-1} u^{-b} \ \log^\Delta u = b^{-1} u^{-b} \ \log^{\gamma-1} u, \ u \to \infty; \tag{2.18}$$

but the upper estimation for the tail function gives only the inequality

$$T_{\beta^{(\Delta)}}(a,u) \leq C_2(b,\gamma) \ u^{-b} \ \log^\gamma u, \ u \geq e. \tag{2.19}$$

Let us show now that the inequality (2.18) is asymptotically exact as $u \to \infty$, by virtue of the consideration of a following example.

Let us denote

$$X(k) = \text{Ent}\left[e^{e^k}\right],$$

where $\text{Ent}[z]$ denotes the integer part of the variable z;

$$p(k) := \exp\left(\gamma bk - b\exp(k)\right),$$

where

$$\gamma = \text{const} > 0, \ b = \text{const} > 1.$$

We define the weight sequence $\beta(k)$ as follows:

$$\beta(X(k)) = p(k)$$

and

$$\beta(l) = 0, \ l \neq X(k) \ \forall k = 1, 2, \ldots.$$

We introduce also the sequence $y(n) = n, n = 1, 2, \ldots.$ It is easy to compute analogously to [25]:

$$|y|_{p,\beta} = \left\{ \sum_k k^p \beta(k) \right\}^{1/p} \asymp (b-p)^{-\gamma}, \ p \in (1,b),$$

but we observe that for the *subsequence* $X(k)$

$$T_\beta(y, X(k)) = T_\beta(y, X(k)) \geq C_1(\gamma, b) \, [\log X(k)]^\gamma \cdot X(k)^{-b}. \qquad (2.20)$$

Note that the "continuous case" was investigated in [25], [31].

EXAMPLE 2. We know that for the sequence $a(n) = n^{-1/L}, L = \text{const} > 1$

$$|a|_p^p = \psi_{a(L)}^p(p) \sim \frac{p}{p-L}, \ p > L;$$

therefore we obtain from the upper tail estimation

$$T(a, \epsilon) \leq C \, \epsilon^{-L} \, |\log \epsilon|, \ \epsilon \in (0, 1/e);$$

but really

$$T(a, \epsilon) \asymp \epsilon^{-L}, \ \epsilon \to 0+.$$

In the more general case when the sequence $a(n)$ has a view

$$a(n) = n^{-1/L} \, \log^q n, L > 1, q > 0,$$

we have:

$$|a|_p^p \sim \left[\frac{p}{p-L} \right]^{pq+1}, \ p > L;$$

$$T(a, \epsilon) \sim \epsilon^{-L} \, |\log \epsilon|^{qL}, \epsilon \to 0+;$$

Note that it follows from the upper estimation only the inequality

$$T(a, \epsilon) \leq C\epsilon^{-L} \, |\log \epsilon|^{1+qL}, \epsilon \in (0, 1/e).$$

But we can show that the our upper bound for the tail function is non-improvable. Namely, in the article [31] was constructed for all the values $L = \text{const} > 1, \ q \geq 0$ the example of *discrete* function $z = z(k), \ k = 1, 2, \ldots$ and the correspondent weight $\beta = \beta(k)$, for which

$$|z|_{p,\beta}^p = \psi_\beta^p(z, p) \sim C_3(L, q) \left[\frac{p}{p-L} \right]^{pq+1}, \ p > L \qquad (2.21)$$

and simultaneously for some positive *subsequence* $\epsilon(m)$ monotonically tending to zero

$$T_\beta(z, \epsilon(m)) \geq C(L, q) \, \epsilon(m)^{-L} \, |\log \epsilon(m)|^{1+qL}, \epsilon(m) \in (0, 1/e). \qquad (2.22)$$

D. LEINDLER'S INEQUALITY FOR DISCRETE GLS SPACES.

Let $\beta = \{\beta(n)\}, n = 1, 2, \ldots$ be again a discrete weight. We introduce a two linear operators:

$$T[x](n) = \sum_{k=n}^{\infty} \frac{x(k)\beta(k)}{\Sigma(k)},$$

where

$$\Sigma(k) = \sum_{j=1}^{k} \beta(j);$$

$$U[x](n) = \sum_{k=1}^{n} \frac{x(k)\beta(k)}{\sigma(k)},$$

where

$$\sigma(k) = \sum_{i=k}^{\infty} \beta(i),$$

in the case when

$$\sum_{i=1}^{\infty} \beta(i) < \infty.$$

Suppose that for some function $\psi \in \Psi$

$$x \in G_d(\psi);$$

for instance ψ may be the natural function for the sequence $x(\cdot): \ \psi(p) = \psi_x(p)$.

Denote

$$\psi_1(p) = p\psi(p).$$

Theorem L. (Leindler's inequality for discrete GLS spaces.)
A.

$$||T[x]||G_d(\psi_1) \le 1 \cdot ||x||G_d(\psi), \tag{2.23}$$

where the constant "1" is the best possible.

B.

$$||U[x]||G_d(\psi_1) \le 1 \cdot ||x||G_d(\psi), \tag{2.24}$$

where the constant "1" is the best possible.

Note that Leindler's inequalities for discrete GLS spaces are used for obtaining the L_p weight estimations for trigonometric series, see [44].

Proof of the upper estimate.

0. We will use the Leindler inequalities [21], (which are some generalizations of the classical Hardy-Littlewood inequalities):

$$\sum_{n=1}^{\infty} \beta(n) \left(\sum_{k=1}^{n} \alpha(k) \right)^p \leq p^p \cdot \sum_{n=1}^{\infty} \beta^{1-p}(n) \alpha^p(n) \left(\sum_{k=n}^{\infty} \beta(k) \right)^p; \qquad (2.25)$$

$$\sum_{n=1}^{\infty} \beta(n) \left(\sum_{k=n}^{\infty} \alpha(k) \right)^p \leq p^p \cdot \sum_{n=1}^{\infty} \beta^{1-p}(n) \alpha^p(n) \left(\sum_{k=1}^{n} \beta(k) \right)^p; \qquad (2.26)$$

In this inequalities $\alpha(n)$ is arbitrary non-negative sequence, $1 \leq p < \infty$.

1. Let us prove the assertion **A** of our theorem; the second may be proved analogously. Note that we can assume that all the variables $x(n)$ are non-negative.

We substitute in (2.25)

$$\alpha(n) = \frac{\beta(n)}{\Sigma(n)} \cdot x(n),$$

where

$$x(\cdot) \in G_d(\psi),$$

as long as in other case is nothing to prove.

We can and will suppose without loss of generality that $||x||G_d(\psi) \leq 1$, or equally

$$\forall p \implies |x|_{p,\beta} \leq \psi(p).$$

The right-hand side $R_a^{(p)}$ of inequality(2.25) has a view:

$$R_a^{(p)} = p^p \, ||x||_{p,\beta}^p,$$

but the left-hand side $L_a^{(p)}$ of this inequality may be rewritten as follows:

$$L_a^{(p)} = ||T[x]||_{p,\beta}^p.$$

We conclude using the first Leindler's inequality

$$||T[x]||_{p,\beta} \leq p \cdot ||x||_{p,\beta} = p \, \psi(p) = \psi_1(p),$$

and after using the direct definition of the norm in GLS spaces,

$$||T[x]||G_d\psi_1 \leq ||x||G_d\psi.$$

Proof of the exactness.

We describe here the method of the lower estimations which will be used often further.

Let us denote

$$V(x; \beta, \psi) = \frac{||T[x]||G_d(\psi_1)}{||x||G_d(\psi)}, \qquad (2.27)$$

$$\overline{V} = \sup_{x \neq 0} \sup_{\psi \in \Psi(1,\infty)} \sup_{\beta > 0} V(x; \beta, \psi), \qquad (2.28)$$

and analogously

$$V_0(x; \beta, \psi) = \frac{||U[x]|| G_d(\psi_1)}{||x|| G_d(\psi)}, \tag{2.29}$$

$$\overline{V}_0 = \sup_{x \neq 0} \sup_{\psi \in \Psi(1, \infty)} \sup_{\beta > 0} V_0(x; \beta, \psi). \tag{2.30}$$

From theorem L follows that

$$V \leq 1; \ V_0 \leq 1.$$

It remains to prove the inverse inequalities.

Note first of all that the expression for the value $V(\cdot)$ may be rewritten as follows:

$$V(x; \beta, \psi) = \frac{\sup_p \left[|T[x]|_{p,\beta} / (p\psi(p))\right]}{\sup_p \left[|x|_{p,\beta} / \psi(p)\right]}, \tag{2.31}$$

and if we choose

$$\psi(p) = |x|_{p,\beta},$$

i.e. $\psi(\cdot)$ is the natural function for the sequence x relative the weight $\beta(\cdot): \ \psi = \psi_x$, we obtain the following lower estimation for the value $\overline{V}$.

Proposition 1.

$$\overline{V} \geq \sup_p W(p), \tag{2.32}$$

where the functional $W = W(p) = W(p; T)$ has (here) a view:

$$W(p) = \sup_{x \neq 0} \sup_{\beta > 0} \left[\frac{|T[x]|_{p,\beta}}{p|x|_{p,\beta}} \right]. \tag{2.33}$$

As a consequence: let x_0 be arbitrary element of the space $l_{p,\beta}$ and β_0 be any sequence satisfying our conditions, then $W(p) \geq W_0(p)$, where

$$W_0(p) = \left[\frac{|T[x_0]|_{p,\beta_0}}{p|x_0|_{p,\beta_0}} \right]. \tag{2.34}$$

Furthermore, if x_0^Δ be arbitrary *set:* $\Delta = $ const of elements of the space $l_{p,\beta}$ and β_0^Δ be any set of the sequences satisfying our conditions, then $W(p) \geq W_1(p)$, where

$$W_1(p) = \sup_\Delta \left[\frac{|T[x_0^\Delta]|_{p,\beta_0^\Delta}}{p|x_0^\Delta|_{p,\beta_0^\Delta}} \right] \tag{2.35}$$

and consequently

$$W_1(p) \geq W_2(p) \stackrel{def}{=} \overline{\lim}_{\Delta \to \infty} \left[\frac{|T[x_0^\Delta]|_{p,\beta_0^\Delta}}{p|x_0^\Delta|_{p,\beta_0^\Delta}} \right] \tag{2.36}$$

Note in addition if there is some value p_0, $p_0 \in \operatorname{supp} \psi$ or the point $p_0 = \infty$, in the case when $\operatorname{supp} \psi = (A, \infty)$, which will be called *critical point,* then

$$\overline{V} \geq \overline{\lim}_{p \to p_0 \pm 0} W(p) \geq \overline{\lim}_{p \to p_0 \pm 0} W_1(p) \geq \overline{\lim}_{p \to p_0 \pm 0} W_2(p). \qquad (2.37)$$

We return to the proof of assertion of the considered theorem L.

Taking here as an examples the values

$$\beta(n) = n^s, \ \alpha(n) = n^{-1-\theta},$$

where $s, \theta = \operatorname{const} > 0$, and

$$p_0 \stackrel{def}{=} (s+1)/\theta > 1,$$

we obtain after simple calculations:

$$\sigma(n) = \sum_{k=n}^{\infty} k^{-1-s} \sim n^{-s}/s, \ n \to \infty;$$

$$x(n) = \alpha(n)\sigma(n)/\beta(n) \sim n^{1+\theta}/s,$$

and we have as $p \to p_0 - 0$:

$$|x|_{p,\beta}^p \sim s^{-p} \sum_{n=1}^{\infty} n^{-1-s+p(1+\theta)} \sim \frac{s^{-p}}{s - p(1+\theta)}.$$

Further,

$$T[x](n) = \sum_{k=1}^{n} k^{\theta} \sim n^{\theta+1}/(\theta+1), \ n \to \infty;$$

$$p \to p_0 - 0 \ \Rightarrow$$

$$|T[x]|_{p,\beta}^p \sim \sum_{n=1}^{\infty} (\theta+1)^{-p} n^{-1-s+p(\theta+1)} \sim \frac{(\theta+1)^{-p}}{s - p(1+\theta)}.$$

Thus,

$$\overline{V} \geq \overline{\lim}_{p \to p_0 - 0} \frac{s}{\theta+1} \cdot \frac{1}{p} = \lim_{p \to p_0 - 0} \frac{p_0}{p} = 1.$$

Analogously may be proved the estimate $\overline{V}_0 \geq 1$. It is sufficient to choose

$$\beta(n) = n^t, \ \alpha(n) = n^{-1-\tau}, \ p_0 = (t+1)/\tau > 1,$$

and

$$\overline{V}_0 \geq \overline{\lim}_{p \to p_0 + 0} \frac{t+1}{\tau} \cdot \frac{1}{p} = 1.$$

4. Exponential Orlicz Spaces.

We will prove in this subsection that the so-called Exponential Orlicz Spaces (EOS) are particular cases of Grand Lebesgue Spaces.

In the case of finite measurable spaces, for example, for the probabilistic spaces this assertion was proved in [18]; see also [24], chapter 1, section 5.

Let $N = N(u)$ be some $N - $ Orlicz's function, i.e. downward convex, even, continuous differentiable for all sufficiently greatest values u, $u \geq u_0$, strongly increasing in the right - side axis, and such that $N(u) = 0 \Leftrightarrow u = 0$; $u \to \infty \Rightarrow dN(u)/du \to \infty$. We say that $N(\cdot)$ is an Exponential Orlicz Function, briefly: $N(\cdot) \in EOF$, if $N(u)$ has a view: for some continuous differentiable strongly increasing downward convex in the domain $[2, \infty]$ function $W = W(u)$ such that $u \to \infty \Rightarrow W^{/}(u) \to \infty$

$$N(u) = N(W, u) = \exp(W(\log |u|)), \ |u| \geq e^2. \tag{2.38}$$

For the values $u \in [-e^2, e^2]$ we define $N(W, u)$ arbitrary but so that the function $N(W, u)$ is even continuous convex strictly increasing in the right side axis and such that $N(u) = 0 \Leftrightarrow u = 0$. The correspondent Orlicz space on T^d, R^d with usually Lebesgue measure with $N - $ function $N(W, u)$ we will denote $L(N) = EOS(W)$; $EOS = \cup_W \{EOS(W)\}$ (Exponential Orlicz's Space).

For example, let $m = \text{const} > 0$, $r = \text{const} \in R^1$,

$$N_{m,r}(u) = \exp \left[|u|^m \left(\log^{-mr}(C_1(r) + |u|) \right) \right] - 1, \tag{2.39}$$

$C_1(r) = e, \ r \leq 0$; $C_1(r) = \exp(r), \ r > 0$. Then $N_{m,r}(\cdot) \in EOS$. In the case $r = 0$ we will write $N_m = N_{m,0}$.

Recall here that the Orlicz's norm on the arbitrary measurable space (X, A, μ) $||f||L(N) = L(N, X, \mu)$ may be calculated by the formula (see, for example, [19],p. 73; [34], p. 66)

$$||f||L(N) = \inf_{v>0} \left\{ v^{-1} \left(1 + \int_X N(v|f(x)|) \ \mu(dx) \right) \right\}. \tag{2.40}$$

Recall also that the notation $N_1(\cdot) << N_2(\cdot)$ for two Orlicz functions N_1, N_2 denotes:

$$\forall \lambda > 0 \ \Rightarrow \lim_{u \to \infty} N_1(\lambda u)/N_2(u) = 0. \tag{2.41}$$

We will denote for arbitrary Orlicz $L(N)$ (and other r.i.) spaces by $L^0(N)$ the closure of all bounded functions with bounded support.

Let α be arbitrary number, $\alpha = \text{const} \geq 1$, and $N(\cdot) \in EOS(W)$ for some $W = W(\cdot)$. We denote for such a function $N = N(W, u)$ by $N^{(\alpha)}(u)$ a new $N - $ Orlicz's function such that

$$N^{(\alpha)}(u) = C_1 |u|^\alpha, \quad |u| \in [0, C_2];$$
$$N^{(\alpha)}(u) = C_3 + C_4|u|, \quad |u| \in (C_2, C_5];$$
$$N^{(\alpha)}(u) = N(u), \quad |u| > C_5, \quad 0 < C_2 < C_5 < \infty, \tag{2.42}$$
$$C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot)).$$

In the case $\alpha = m(j + 1)$, $m > 0$, $j = 0, 1, 2, \ldots$ the function $N_m^{(\alpha)}(u)$ is equivalent to the following Trudinger's function:

$$N_m^{(\alpha)}(u) \sim N_{[m]}^{(\alpha)}(u) = \exp \left(|u|^m \right) - \sum_{l=0}^{j} u^{ml}/l!. \tag{2.43}$$

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This method is described in [34], p. 42 - 47. Those Orlicz spaces are applied to the theory of non - linear partial differential equations.

We can define formally the spaces $L(N_m^{(\alpha)})$ at $m = +\infty$ as a projective limit at $m \to \infty$ the spaces $L(N_m^{(\alpha)})$, but it is evident that

$$L\left(N_\infty^{(\alpha)}\right) \sim L_\alpha + L_\infty,$$

where the space L_∞ consists on all the a.e. bounded functions with norm

$$|f|_\infty = \operatorname*{vraisup}_{x \in X} |f(x)|.$$

Of course, in the case $X = T^d$

$$L_\alpha + L_\infty \sim L_\infty.$$

HEREAFTER WE WILL DENOTE BY $C_k = C_k(\cdot), k = 1, 2, \ldots$ SOME POSITIVE FINITE ESSENTIALLY AND BY C, C_0 NON-ESSENTIALLY "CONSTRUCTIVE" CONSTANTS.

BY THE SYMBOLS $K_j = K_j(d)$ WE WILL DENOTE THE "CLASSICAL" *absolute* CONSTANTS; MORE EXACTLY, POSITIVE FINITE FUNCTIONS DEPENDING ONLY ON THE DIMENSION d.

It is very simple to prove the existence of constants $C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot))$ such that $N^{(\alpha)}$ is some new exponential N Orlicz's function.

Now we will introduce some *new* Grand Lebesgue Spaces. Let $\psi = \psi(p)$, $p \geq \alpha, \alpha = \text{const} \geq 1$ be some continuous positive: $\psi(\alpha) > 0$ finite strictly increasing function such that the function $p \to p \log \psi(p)$ is downward convex and

$$\lim_{p \to \infty} \psi(p) = \infty.$$

The set of all those functions we will denote Ψ; $\Psi = \{\psi\}$. A particular case:

$$\psi(p) = \psi(W; p) = \exp(W^*(p)/p),$$

where

$$W^*(p) = \sup_{z \geq \alpha}(pz - W(z)) \qquad (2.44)$$

is so - called Young - Fenchel, or Legendre transform of $W(\cdot)$. It follows from theorem of Fenchel - Moraux that in this case

$$W(p) = [p \ \log \psi(W; p)]^*, \quad p \geq p_0 = \text{const} \geq 2, \qquad (2.45)$$

and consequently for all $\psi(\cdot) \in \Psi$ we introduce the correspondent $N -$ function by equality:

$$N([\psi]) = N([\psi], u) = \exp\left\{[p \log \psi(p)]^* (\log u)\right\}, \ u \geq e^2. \qquad (2.46)$$

Since $\forall \ \psi(\cdot) \in \Psi, \ d = 0, 1, \ldots \Rightarrow p^d \cdot \psi(p) \in \Psi$, we can denote

$$\psi_d(p) = p^d \cdot \psi(p), \quad N_d([\psi]) = N_d([\psi], u) = N([\psi_d], u).$$

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For instance, if $N(u) = \exp(|u|^m),\ u \geq 2,$ where $\ m = \text{const} > 0,$ then

$$N_d([\psi], u) \sim \exp\left(|u|^{m/(dm+1)}\right),\ u \geq 2.$$

Definition. We introduce for arbitrary such a function $\psi(\cdot) \in \Psi$ the so - called $G(\alpha; \psi)$ and $G(\alpha; \psi, \nu)$ norms and correspondent Banach spaces $G(\alpha; \psi),\ G(\alpha, \psi, \nu)$ as a set of all measurable (complex) functions with finite norms:

$$||f||G(\alpha; \psi) = \sup_{p \geq \alpha}(|f|_p/\psi(p)), \qquad (2.47)$$

and analogously

$$||f||G(\alpha; \psi, \nu) = \sup_{p \geq \alpha}(|f|_p(\nu)/\psi(p)). \qquad (2.48)$$

For instance $\psi(p)$ may be $\psi(p) = \psi_m(p) = p^{1/m},\ m = \text{const} > 0;$ in this case we will write $G(\alpha, \psi_m) = G(\alpha, m)$ and

$$||f||G(\alpha, m) = \sup_{p \geq \alpha}\left(|f|_p\, p^{-1/m}\right). \qquad (2.49)$$

Also formally we define

$$||f||G(\alpha, m) = |f|_\alpha + |f|_\infty.$$

Remark 1. It follows from I'ensen inequality that in the case $X = T^d$ all the spaces $G(\alpha_1; \psi),\ G(\alpha_2, \psi), 1 \leq \alpha_1 < \alpha_2 < \infty$ are isomorphic:

$$||f||G(\alpha; \psi) \leq ||f||G(1; \psi) \leq \max(1, \psi(\alpha))\, ||f||G(\alpha; \psi). \qquad (2.50)$$

It is false in the case $X = R^d$.

Remark 2. $G(\alpha; \psi)$ is a rearrangement invariant (r.i.) space. $G(\alpha, m)$ has a fundamental function $\phi(\delta; G(\alpha, m)),\ \delta > 0,$ where for any rearrangement invariant space G

$$\phi(\delta; G) \stackrel{def}{=} ||I_A(\cdot)||G(\cdot),\ \ \text{mes}(A) = \delta \in (0, \infty),$$

$\text{mes}(A)$ denotes usually Lebesgue measure of Borel set A. We have:

$$\phi(\delta; G\psi) = \sup_{p}[\delta/\psi(p)], \delta \in (0, \infty).$$

The detail investigating of $G\psi$ spaces, for instance, their fundamental functions see in [22], [25].

Let us consider also another space $G(a, b, \alpha, \beta),\ 1 \leq a < b < \infty;\ \alpha, \beta \geq 0.$ Here $X = R^d$ and we denote $h = \min((a+b)/2; 2a).$ We introduce the function $\zeta : (a, b) \to R_+^1 :$

$$\zeta(p) = \zeta(a, b, \alpha, \beta; p) = (p - a)^\alpha,\ p \in (a, h);$$

$$\zeta(p) = (b - p)^\beta,\ p \in [h, b).$$

By definition, the space $G(a, b, \alpha, \beta)$ consists on all the measurable complex functions with finite norm:

$$||f||G(a, b, \alpha, \beta) = \sup_{p \in (a,b)} \left[|f|_p \cdot \zeta(a, b, \alpha, \beta; p) \right].$$

The space $G(a, b, \alpha, \beta)$ is also a rearrangement invariant space.

For example, let us consider the function $f(x) = f(a, b; x), \ x \in R^1 \to R :$ $f(x) = 0, \ x \le 0;$

$$f(x) = x^{-1/b}, \ x \in (0, 1); \quad f(x) = x^{-1/a}, \ x \in [1, \infty);$$

then $f(a, b, \cdot) \in G(a, b, 1, 1)$ and

$$\forall \, \Delta \in (0, 1/2] \ \Rightarrow f \notin G(a, b, 1 - \Delta, 1) \cup G(a, b, 1, 1 - \Delta).$$

Analogously may be defined the "discrete" $g(a, b, \alpha, \beta)$ spaces. Namely, let $c = c(n) = c(n_1, n_2, \ldots, n_d)$ be arbitrary multiply (complex) sequence. We say that $c \in g(a, b, \alpha, \beta)$ if

$$||c||g(a, b, \alpha, \beta) \overset{def}{=} \sup_{p \in (a,b)} \left[|c|_p \ (p - a)^\alpha \ (b - p)^\beta \right].$$

It is evident that the non - trivial case of those spaces is only if $\beta = 0$; in this case we will write $g(a, b, \alpha, 0) = g(a, \alpha)$ and

$$||c||g(\alpha) = \sup_{p > a} |c|_p \ (p - a)^\alpha.$$

We denote also for $\psi(\cdot) \in \Psi : \ ||c||g(\psi, \nu) =$

$$\sup_{p \ge 2} \left[|c|_p(\nu)/\psi(p) \right], \quad ||c||_m(\nu) = \sup_{p \ge 2} \left[|c|_p(\nu) \ \cdot p^{-1/m} \right], \quad m = \text{const} > 0.$$

Note than our Orlicz N − functions $N \in EOS$ does not satisfy the so-called Δ_2 condition.

3 Boundedness of Hilbert's transform in GLS

We consider in this section the case $T = [-\pi, \pi]$ equipped with the classical Lebesgue measure, i.e. $d = 1$, and the case of Hilbert's transform in GLS spaces.

Recall that for the integrable function $f : T \to R$ with the correspondent Fourier series

$$f(x) = 0.5a(0) + \sum_{k=1}^{\infty} \left[a(k) \cos kx + b(k) \sin kx \right], \tag{3.1}$$

where as ordinary

$$a(k) = \pi^{-1} \int_T f(t) \cos(kt) dt, \ b(k) = \pi^{-1} \int_T f(t) \sin(kt) dt,$$

the Hilbert's transform $H[f](x)$ may be defined as follows:

$$H[f](x) = \sum_{n=1}^{\infty} [a(n) \sin(nx) - b(n) \cos(nx)].$$

Equivalent definition:

$$H[f](x) = (2\pi)^{-1} \ p.v. \int_T f(x - t) \cot(t/2) \ dt. \qquad (3.2)$$

See in detail, e.g., the classical monograph of A.Zygmund [48], chapter 11.

Let $p \in (1, \infty)$. It is known that the operator $H[f]$ is bounded in all the spaces $L_p = L_p(T)$. The *exact value* of the norm

$$K_H(p) \stackrel{def}{=} \sup_{f \in L_p, f \neq 0} |H[f]|_p / |f|_p = |H|(L_p \to L_p)$$

was computed by S.K.Pichorides [32]:

$$K_H(p) = \tan(\pi/(2p)), \ p \in (1, 2]; \ K_H(p) = \cot(\pi/(2p)), \ p \in [2, \infty). \qquad (3.3)$$

Let now $\psi(\cdot) \in \Psi(1, \infty)$, i.e. $\text{supp} \, \psi \subset (1, \infty)$. We define the new ψ function $\psi^{(H)}(p), \ p \in (1, \infty)$ as follows:

$$\psi^{(H)}(p) = K_H(p) \cdot \psi(p), \ p \in (1, \infty). \qquad (3.4)$$

Theorem H.

$$||H[f]||G\psi^{(H)} \leq 1 \cdot ||f||G\psi, \qquad (3.5)$$

where the constant "1" is the best possible.

Proof of the upper bound is very simple. Let $||f||G\psi < \infty$, since in other case is nothing to prove. Moreover, we can and will suppose $||f||G\psi = 1$, or following

$$\forall p \in (1, \infty) \ \Rightarrow |f|_p \leq \psi(p).$$

It follows from the Pichorides inequality

$$|H[f]|_p \leq K_H(p) \cdot |f|_p \leq K_H(p) \cdot \psi(p) = \psi^{(H)}(p),$$

therefore

$$||H[f]||G\psi^{(H)} \leq 1 = ||f||G\psi.$$

Proof of the exactness. We will use the method of the proposition 1. Let us denote

$$V(f, \psi) = \frac{||H[f]|| \, |G\psi^{(H)}}{||f|| \, |G\psi} = \frac{\sup_{p>1}[|H[f]|_p/(K_H(p) \cdot \psi(p))]}{\sup_{p>1}[|f|_p/\psi(p)]},$$

$$\overline{V} = \sup_{\psi \in \Psi} \sup_{f \neq 0, f \in G\psi} V(f, \psi).$$

The assertion of theorem H may be formulated as equality $\overline{V} = 1$; we proved $\overline{V} \leq 1$. It remains to prove that $\overline{V} \geq 1$.

If we implement the natural choice of the function $\psi(p)$ for the $f(\cdot)$: $\psi(p) = |f|_p$, we receive the inequality

$$\overline{V} \geq \sup_{p>1} \sup_{f \in L_p} \left[\frac{|H[f]|_p}{K_H(p) \cdot |f|_p} \right] \geq \overline{\lim}_{p \to \infty} \sup_{f \in L_p} \left[\frac{|H[f]|_p}{K_H(p) \cdot |f|_p} \right].$$

Let us consider the *family* of a functions

$$g_\Delta(x) = \sum_{n=1}^{\infty} n^{-1} \, \log^\Delta(n) \, \sin nx, \quad \Delta = \text{const} > 0.$$

It is known, see, e.g., [48], chapter 8, that as $x \to 0$

$$g_\Delta(x) \sim \frac{2}{\pi} |\log |x||^\Delta,$$

therefore as $p \to \infty$

$$|g_\Delta(\cdot)|_p \sim 2^{1/p} \frac{2}{\pi} \left[\int_0^\pi |\log x|^\Delta dx \right]^{1/p} \sim$$

$$2^{1/p} \frac{2}{\pi} (\Gamma(\Delta p + 1))^{1/p} \sim 2^{1/p} \frac{2}{\pi} \left[\frac{\Delta p}{e} \right]^\Delta.$$

Further,

$$f_\Delta(x) := H[g_\Delta](x) = \sum_{n=1}^{\infty} n^{-1} \, \log n^\Delta \, \cos(nx),$$

then as $x \to 0$

$$|f_\Delta(x)| \sim \frac{|\log |x||^\Delta}{\Delta + 1},$$

and correspondingly as $p \to \infty$, i.e. the critical point $p_0 = \infty$:

$$|f_\Delta(\cdot)|_p \sim 2^{1/p} (\Delta + 1)^{-1} \left[\frac{(\Delta + 1)p}{e} \right]^{\Delta + 1};$$

$$\frac{|f_\Delta(\cdot)|_p}{|g_\Delta(\cdot)|_p} \sim \frac{2}{\pi} \cdot p \, e^{-1} \left(1 + \frac{1}{\Delta} \right)^\Delta.$$

It follows from the Pichorides result that as $p \to \infty$

$$K_H(p) \sim \frac{2}{\pi} \, p.$$

We find substituting into the expression for $\overline{V}$ for all the values $\Delta > 0$:

$$\overline{V} \ge e^{-1} \left(1 + \frac{1}{\Delta}\right)^{\Delta}.$$

The expression in the right hand side tends to one as $\Delta \to \infty$.

This completes the proof of our theorem.

Analogous result is true for "continuous" Hilbert's transform, i.e. in the space $X = R^1$. Recall that in this case

$$H[f](x) = \pi^{-1} \; p.v. \int_{-\infty}^{\infty} \frac{f(t) \, dt}{x - t}. \tag{3.6}$$

Since

$$K_H(p) \stackrel{def}{=} |H|(L_p \to L_p) \sim \frac{2}{\pi} \frac{1}{p-1}, \; p \to 1 + 0,$$

the correspondent example may be constructed as follows:

$$f_0(x) = 1, \; x \in (0, 1), \; f_0(x) = 0, \; x < 0, \; x > 1;$$

then

$$H[f_0](x) = \pi^{-1} \log \left| \frac{x}{x - 1} \right|;$$

$$x \to \infty \; \Rightarrow \pi H[f_0](x) \sim 1/x;$$

$$|\pi H[f_0]|_p \sim \frac{2}{\pi} \frac{1}{p-1} = \frac{2}{\pi} \frac{1}{p-1} |f_0|_p, \; p \to 1 + 0. \tag{3.7}$$

See in detail [2], p. 126-128.

4 Weight Fourier's inequalities for GLS spaces

Let again $d = 1, \; X = [-\pi, \pi]$,

$$f(x) = 0.5a(0) + \sum_{k=1}^{\infty} [a(k) \cos kx + b(k) \sin kx],$$

$$\gamma = \text{const} \in (0, 1), U_\gamma[f] = |x|^{-\gamma} f(x),$$

and we define the sequence $\lambda(n), n = 1, 2, \ldots$ as follows:

$$\lambda(1) = a(0), \lambda(2) = b(1), \lambda(3) = a(1), \lambda(4) = b(2), \lambda(5) = a(2), \ldots.$$

We intend to obtain in this section the GLS norm estimation for the function $U_\gamma[f]$ through the GLS norm estimation for the coefficients $\{\lambda(n)\}$.

We consider in this section that both the sequences $a(n)$ and $b(n)$ are monotonically decreasing; more general case may be investigated by means of the main result of the article [44], see also [1].

A new notations: $p_0 = 1/\gamma$, (critical point);

$$|\vec{\lambda}|_p^{(\gamma)} = \left[\sum_{n=1}^{\infty} n^{p(1+\gamma)-2}|\lambda(n)|^p\right]^{1/p}, \tag{4.1}$$

$$l_p^{(\gamma)} = \{\vec{\lambda}: \ |\vec{\lambda}|_p^{(\gamma)} < \infty\}, \tag{4.2}$$

and we define for arbitrary function $\psi \in G\Psi(1, p_0)$

$$||\vec{\lambda}||G^{(\gamma)}\psi = \sup_{p \in (1,p_0)} |\vec{\lambda}|_p^{(\gamma)}/\psi(p), \tag{4.3}$$

$$\psi^{(\gamma)}(p) = K^{(\gamma)}(p) \cdot \psi(p), \ p \in (1, p_0); \tag{4.4}$$

where

$$K^{(\gamma)}(p) \stackrel{def}{=} \sup_{\lambda \in l_p^{(\gamma)}, \lambda \neq 0} \left[\frac{|U_\gamma[f]|_p}{|\vec{\lambda}|_p^{(\gamma)}}\right] < \infty, \ p \in (1, p_0). \tag{4.5}$$

Theorem γ.

$$||f||G\psi^{(\gamma)} \leq 1 \cdot ||\vec{\lambda}||G^{(\gamma)}\psi,$$

where the constant "1" is the best possible.

Proof.

1. It follows after some calculations in the article [44], see also [3], [5], [6] that

$$K^{(\gamma)}(p) \leq \frac{C_\gamma(p) \ \gamma^{-2}}{1/\gamma - p}, \tag{4.6}$$

where $C_\gamma(p)$ is continuous function of the variable p on the *closed* interval $p \in [1, p_0]$, and such that

$$\lim_{p \to p_0 - 0} C(p) = 1.$$

2. Let us estimate from below the constant $K^{(\gamma)}(p)$. It is enough to consider the following example.

$$g_\Delta(x) = \sum_{n=2}^{\infty} n^{-1} \ \log^\Delta(n) \ \cos(nx); \ \Delta = \text{const} > 0, \tag{4.7}$$

i.e.here $a(n) = 0, \ n = 0, 1, 2, \ldots; \ \lambda(n) = b(n) = n^{-1}\log^\Delta(n)$.

It is easy to calculate as $p \to p_0 - 0$:

$$\sum_{n=2}^{\infty} \left[n^{p\gamma-2} \frac{\log^{p\Delta}(n)}{n^p} \right]^{1/p} \sim \frac{\Gamma^\gamma(\Delta/\gamma+1)}{(1-p\gamma)^{\Delta+\gamma}};$$

$$g_\Delta(x) \sim (\Delta+1)^{-1} \, |\log|x| \, |^{\Delta+1}, \ x \to 0;$$

$$|g_\Delta|_p^{(\gamma)} \sim (\Delta+1)^{-1} \frac{\Gamma^\gamma((\Delta+1)/\gamma)}{(1-p\gamma)^{(1+\Delta)/\gamma+1}}, \ p \to p_0 - 0,$$

and we obtain after dividing as $p \to p_0 - 0$

$$\overline{K} := (1-p\gamma) \, \frac{|g_\Delta|_p^{(\gamma)}}{|\lambda|_p^{(\gamma)}} \sim \frac{\Gamma^\gamma((1+\Delta)+1)}{(\Delta+1)\Gamma^\gamma(\Delta/\gamma+1)}.$$

It follows from Stirling's formula that as $\Delta \to \infty$

$$\overline{K} \sim e^{-1}\gamma^{-1} \left[\frac{1+\Delta}{\Delta} \right]^\Delta \to \gamma^{-1}.$$

So, one has as $p \to p_0 - 0$:

$$K^{(\gamma)}(p) \sim \frac{\gamma^{-1}}{1-p\gamma} = \frac{\gamma^{-2}}{p_0-p}. \tag{4.7}$$

As before, the last assertion proves the proposition of the considered theorem.

5   Boundedness of Fourier's transform

Theorem 1. *Let* $X = [0, 2\pi]^d$ *and* $\psi \in \Psi$. *Then the Fourier operators* $s_M[\cdot]$ *are uniformly bounded in the space* $L(N[\psi])$ *into another exponential Orlicz's space* $L(N_d[\psi])$:

$$\sup_{M \geq 1} ||s_M[f]||L(N_d[\psi]) \leq C_6(d, \psi) \, ||f||L(N[\psi]). \tag{5.1}$$

Theorem 2. *Let now* $X = R^d$, $\psi \in \Psi$ *and* $\alpha = \text{const} > 1$. *The Fourier operators* $S_M[\cdot]$ *are uniformly bounded in the space* $L(N^{(\alpha)}[\psi])$ *into the space* $L(N_d^{(\alpha)}[\psi])$:

$$\sup_{M \geq 1} ||S_M[f]||L(N_d^{(\alpha)}[\psi]) \leq C_7(\alpha, d, \psi) \, ||f||L(N^{(\alpha)}[\psi]). \tag{5.2}$$

Since the function $N[\psi]$ does not satisfies the Δ_2 condition, the assertions (5.1) and (5.2) does not mean that in general case when $f \in L(N_d^\alpha[\psi])$

$$\lim_{M \to \infty} ||s_M[f] - f||L(N_d[\psi]) = 0, \tag{5.3}$$

$$\lim_{M \to \infty} ||S_M[f] - f||L(N_d^{(\alpha)}[\psi]) = 0; \tag{5.4}$$

see examples further. But it is evident that propositions (5.3) and (5.4) are true if correspondingly

$$f \in L^0(N_d[\psi]), \quad f \in L^0(N_d^{(\alpha)}[\psi]).$$

Also it is obvious that if $f \in L(N_d[\psi]), X = [0, 2\pi]^d$ or, in the case $X = R^d$, $f \in L(N_d^{(\alpha)}[\psi])$, then for all EOF $\Phi(\cdot)$ such that $\Phi << N_d[\psi]$ or $\Phi << N_d^{(\alpha)}([\psi])$ the following implications hold:

$$\forall f \in L(N_d[\psi]) \Rightarrow \lim_{M \to \infty} ||s_M[f] - f||L(\Phi) = 0, \ X = [0, 2\pi]^d; \qquad (5.5)$$

$$\forall f \in L(N_d[\psi]) \Rightarrow \lim_{M \to \infty} ||S_M[f] - f||L(\Phi) = 0, \ X = R^d. \qquad (5.6)$$

Theorem 3. *Let $\Phi(\cdot)$ be an EOF and let $N(\cdot) = L^{-1}(u) \in EOF$, where $L(y), \ y \geq \exp(2)$ is a positive slowly varying at $u \to \infty$ strongly increasing continuous differentiable in the domain $[\exp(2), \infty)$ function such that the function*

$$W(x) = W_L(x) = \log L^{-1}(\exp x), \quad x \in [2, \infty)$$

is again strong increasing to infinity together with the derivative dW/dx. In order to the implication (5.5) or, correspondingly, (5.6) holds, it is necessary and sufficient that $\Phi << L(N_d[\psi])$, or, correspondingly $\Phi << L(N_d^{(\alpha)}[\psi])$.

For instance, the conditions of theorem 3 are satisfied for the functions $N = N_{m,r}(u)$.

Theorem 4. *Let $\psi \in G\Psi(1, 2)$; we denote*

$$\zeta(p) = \psi(p/(p-1)).$$

We assert:

$$||F[f]||G\zeta \leq C(d) \ ||f||G\psi, \ f \in G\psi,$$

and the last estimation is non-improvable.

As a consequence: let $f(\cdot) \in G(1, b, \alpha, 0), \alpha > 0$. Then $F[f] \in L(N_{1/\alpha}^{(2)})$ and

$$\sup_{M \geq 1} ||S_M[f]||L\left(N_{1/\alpha}^{(2)}\right) \leq C_8(\alpha, N) \ ||f||G(1, b, \alpha, 0).$$

Analogously may be formulated (and proved) the "discrete" analog of this result.

Theorem 4a. *Let $\psi \in G_d\Psi(1, 2)$; we denote for the bilateral complex sequence*

$$c = \vec{c} = \{\ldots, c(-2), c(-1), c(0), c(1), c(2), \ldots\}$$

$$F[c](x) = F(x) = \sum_{k=-\infty}^{\infty} c(k) \exp(ikx);$$

$$\zeta_d(p) = \psi_d(p/(p-1)).$$

We assert:

$$||F[f]||G\zeta_d \leq C(d) \ ||c||G\psi_d, \ c \in G_d\psi,$$

and the last estimation is also non-improvable.

Theorem 5 A. *Let $\{\phi_k(x), k = 1, 2, \ldots\}$ be an orthonormal uniform bounded:*

$$M := \sup_k \operatorname{vraisup}_x |\phi_k(x)| < \infty$$

sequence of a functions on some non-trivial measurable space (X, A, μ) and (in the $L_2(X, \mu)$ sense)

$$f(x) = \sum_{k=1}^{\infty} c(k) \ \varphi_k(x). \tag{5.7}$$

A). *If $c \in g(\psi, \nu)$, then*

$$||f||L\left(N_1[\psi], X, \mu\right) \leq C_9 \cdot (1 + M) \cdot ||c||g(\psi, \nu). \tag{5.8}$$

This result may be reformulated as follows. Let $c = \vec{c} = \{c(k)\}, \ k = 1, 2, \ldots$ be some numerical sequence such that for some $\psi \in G_{d,\nu}\Psi \ c \in G_{d,\nu}\psi$. For instance, the function $\psi(p)$ may be natural: $\psi(p) = \psi_0(c, p; \nu)$ for the sequence $\{c(k)\}$ relative the $\nu(\cdot)$ norm:

$$\psi_0(c, p; \nu) := \left[\sum_{k=1}^{\infty} |c(k)|^p \ k^{p-2}\right]^{1/p},$$

if there exists for some non-trivial interval $p \in (A, B); 1 \leq A < B \leq \infty$.

We define also

$$\tilde{\psi} = p \cdot (1 + M) \cdot \psi(p).$$

Theorem 5 A'.

$$||f||G\tilde{\psi} \leq K_3 \ ||c||G_d\psi,$$

and the last inequality is asymptotically exact.

The proof is at the same as in the proposition 1. Note that the point $p = \infty$ is unique "critical" point in this considerations.

Theorem 5 B. *Let $c \in g(\alpha)$ for some $\alpha \in (0, 1]$. We assert that*

$$||f||L\left(N_{1/\alpha}, X, \mu\right) \leq C_{10}(\alpha) \cdot (\max(1, \sup_{k,x} |\phi_k(x)|) \cdot \ ||c(\cdot)||g(\alpha).$$

Theorem 5 C. *Let $\{\phi_k(x), k = 1, 2, \ldots\}$ be again some orthonormal uniform bounded:*

$$M := \sup_k \operatorname{vraisup}_x |\phi_k(x)| < \infty$$

sequence of a functions on some non-trivial measurable space (X, A, μ) *and (in the* $L_2(X, \mu)$ *sense)*

$$f(x) = \sum_{k=1}^{\infty} c(k) \; \varphi_k(x).$$

Let $c = \vec{c} \in G_d \psi$ *for some* $\psi \in \Psi(1, 2)$. *Denote*

$$\tau(q) = (1 + M) \cdot \psi(q/(q-1)), \; q \in (2, \infty).$$

Proposition:

$$||c||G_d \tau \leq C_5 \; ||f||G\psi,$$

and the last inequality is asymptotically exact.

Theorem 6. *If* $f \in G(\alpha; \psi, \nu),$ *where* $\alpha \geq 2,$ *then*

$$\sup_{M \geq 1} ||S_M[f]||L\left(N_d^{(\alpha)}[\psi]\right) \leq C_{11}(\alpha, \psi, N, \nu) \; ||f||G(\alpha; \psi, \nu). \qquad (5.9)$$

6   Auxiliary results

Theorem 7.   *Let* $N(u) = N(W, u) = \exp(W(\log u)), \; u > e^2, \; \psi(p) = \exp(W^*(p)/p), \; p \geq 2,$ *and* $X = T^d$. *We propose that the Orlicz's norm* $|| \cdot ||L(N)$ *and the norm* $|| \cdot ||G(\psi)$ *are equivalent. Moreover, in this case* $f \neq 0, \; f \in G(\psi)$ *(or* $f \in L(N(W(\cdot), u))$ *) if and only if* $\exists C_{12}, C_{13}, C_{14} \in (0, \infty) \Rightarrow \forall \; u > C_{14}$

$$T(|f|, u) \leq C_{12} \exp\left(-W\left(\log\left(u/C_{13}\right)\right)\right), \qquad (6.1)$$

where for each measurable function $f : X \to R$

$$T(|f|, u) = \text{mes}\{x : |f(x)| > u\}.$$

Proof *of theorem 7.*   A). *Assume at first that* $f \in L(N), \; f \neq 0$. *Without loss of generality we suppose that* $||f||L(N) = 1/2$. *Then*

$$\int_X N(W, |f(x)|) \; dx \leq 1 < \infty.$$

The proposition (6.1) follows from Tchebyshev's inequality such that in (6.1) $C_{12} = 1, \; C_{13} = C_{14} = 1/||f||L(N), \; f \neq 0$.

B). Inversely, assume that $f, \; f \neq 0$ is a measurable function, $f : X \to R^1$ such that

$$T(|f|, u) \leq \exp(-W(\log u)), \; u \geq e^2.$$

We have by virtue of properties of the function W :

$$\int_X N(|f(x)|/e^2) \; dx = \int_{\{x:|f(x)| \leq e^2\}} + \int_{\{x:|f(x)| > e^2\}} = I_1 + I_2;$$

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$$I_1 \leq \int_X N(1) \; dx = N(1),$$

$$I_2 \leq \sum_{k=2}^{\infty} \int_{e^k < |f| \leq e^{k+1}} \exp(W(|f(x)|/e^2)) \; dx \leq$$

$$\sum_{k=2}^{\infty} \exp((W(k-1)) \; T(|f|, k) \leq \sum_{k=2}^{\infty} \exp(W(k-1) - W(k)) < \infty.$$

Thus, $f \in L(N(W))$ and

$$\int_X N(|f(x)|/e^2) \; dx \leq N(1) + \sum_{k=2}^{\infty} \exp(W(k-1) - W(k)) < \infty.$$

C). Let now $f \in G(\psi)$; without loss of generality we can assume that $||f||G(\psi) = 1$. We deduce for $p \geq 2$:

$$\int_X |f(x)|^p \; dx \leq \psi^p(p).$$

We obtain using again the Tchebyshev's inequality:

$$T(|f|, u) \leq u^{-p} \psi^p(p) = \exp\left[-p \log u + p \log \psi(p)\right],$$

and after the minimization over p : $u \geq \exp(2) \;\Rightarrow$

$$T(|f|, u) \leq \exp\left(-\sup_{p \geq 2}(p \log u - p \log \psi(p))\right) =$$

$$\exp\left((p \log \psi(p))^*(\log u)\right) = \exp(-W(\log u)).$$

D). Suppose now that $T(|f|, x) \leq \exp(-W(\log x)), \; x \geq \exp(2)$. We conclude:

$$\int_X |f(x)|^p \; dx = p \int_o^{\infty} x^{p-1} T(|f|, x) dx = p \int_0^{\exp(2)} +$$

$$p \int_{\exp(2)}^{\infty} \leq p \int_0^{\exp(2)} x^{p-1} dx + p \int_{\exp(2)}^{\infty} x^{p-1} \; T(|f|, x) dx \leq$$

$$e^{2p} + \int_{\exp(2)}^{\infty} p x^{p-1} \exp(-W(\log x)) \; dx =$$

$$e^{2p} + p \int_2^{\infty} \exp(py - W(y)) \; dy, \; p \geq 2.$$

We obtain using Laplace's method and theorem of Fenchel - Moraux:

$$\int_X |f(x)|^p dx \leq e^{2p} + C^p \exp\left(\sup_{y \geq 2}(py - W(y))\right) = e^{2p} +$$

$$C^p \exp(W^*(p)) = e^{2p} + C^p \exp(p \log \psi(p)) \leq C^p \; \psi^p(p).$$

Finally, $||f||G(\psi) < \infty$.

For example, if $m > 0$, $r \in R$, then

$$f \in L\left(N_{m,r}\right) \Leftrightarrow \sup_{p \geq 2}\left[|f|_p \, p^{-1/m} \, \log^{-r} p\right] < \infty \; \Leftrightarrow$$

$$T(|f|, u) \leq C_0(m, r) \exp\left(-C(m, r)u^m\left(\log^{-mr} u\right)\right), \; u \geq 2.$$

Remark 3. If conversely

$$T(|f|, x) \geq \exp(-W(\log x)), \; x \geq e^2,$$

then for sufficiently large values of p; $p \geq p_0 = p_0(W) \geq 2$

$$|f|_p \geq C_0(W) \, \psi(p), \quad C_0(W) \in (0, \infty). \tag{6.2}$$

Remark 4. In this proof we used only the condition $0 < \mathrm{mes}(X) < \infty$. Therefore, our conclusions in theorem 7 are true in this more general case.

Theorem 8. *Let $\psi \in \Psi$. We assert that $f \in L^0(N[\psi])$, or, equally, $f \in G^0(\psi)$ if and only if*

$$\lim_{p \to \infty} |f|_p / \psi(p) = 0. \tag{6.3}$$

Proof. It is sufficient by virtue of theorem 7 to consider only the case of $G(\psi)$ spaces.

1. Denote $G^{00}(\psi) = \{f : \lim_{p \to \infty} |f|/\psi(p) = 0\}$. Let $f \in G^0(\psi)$, $f \neq 0$. Then for arbitrary $\delta = \mathrm{const} > 0$ there exists a constant $B = B(\delta, f(\cdot)) \in (0, \infty)$ such that

$$||f - fI(|f| \leq B) \, ||G(\psi) \leq \delta/2.$$

Since $|f|I(|f| \leq B)| \leq B$, we deduce

$$|fI(|f| \leq B)|_p / \psi(p) \leq B/\psi(p).$$

We obtain using triangular inequality for sufficiently large values $p : \; p \geq p_0(\delta) = p_0(\delta, B) \; \Rightarrow$

$$|f|_p / \psi(p) \leq \delta/2 + B/\psi(p) \leq \delta,$$

as long as $\psi(p) \to \infty$ at $p \to \infty$. Therefore $G^0(\psi) \subset G^{00}(\psi)$.

(The set $G^{00}(\psi)$ is a closed subspace of $G(\psi)$ with respect to the $G(\psi)$ norm and contains all bounded functions.)

2. Inversely, assume that $f \in G^{00}(\psi)$. We deduce denoting $f_B = f_B(x) = f(x)I(|f| > B)$ for some $B = \mathrm{const} \in (0, \infty)$:

$$\forall Q \geq 2 \; \Rightarrow \lim_{B \to \infty} |f_B|_Q = 0.$$

Further,

$$||f_B||G(\psi) = \sup_{p \geq 2} |f_B|_p / \psi(p) \leq \max_{p \leq Q} |f_B|_p / \psi(p) +$$

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$$\sup_{p>Q} |f_B|_p/\psi(p) \stackrel{def}{=} \sigma_1 + \sigma_2;$$

$$\sigma_2 = \sup_{p>Q} |f_B|_P/\psi(p) \leq \sup_{p\geq Q}(|f|_p/\psi(p)) \leq \delta/2$$

for sufficiently large Q as long as $f \in G^{00}(\psi)$. Let us now estimate the value σ_1 :

$$\sigma_1 \leq \max_{p\leq Q} |f_B|_p/\psi(2) \leq \delta/2$$

for sufficiently large $B = B(Q)$. Therefore,

$$\lim_{B\to\infty} ||f_B||G(\psi) = 0, \quad f \in G^0(\psi).$$

Theorem 9. *Let* $\psi(\cdot) = \psi_N(\cdot), \theta(\cdot) = \theta_\Phi(\cdot)$ *be a two functions on the classes* Ψ *with correspondent* $N - Orlicz's$ *functions* $N(\cdot), \Phi(\cdot)$:

$$N(u) = \exp\left\{[p\log\psi(p)]^* (\log u)\right\},$$

$$\Phi(u) = \exp\left\{[p\log\theta(p)]^* (\log u\right\}, \quad u \geq \exp(2).$$

We assert that $\lim_{p\to\infty} \psi(p)/\theta(p) = 0$ *if and only if* $N(\cdot) >> \Phi(\cdot)$.

    Proof of theorem 9. A). Assume at first that $\lim_{p\to\infty} \psi(p)/\theta(p) = 0$. Denote $\epsilon(p) = \psi(p)/\theta(p)$, then $\epsilon(p) \to 0, \ p \to \infty$.

Let $\{f_\zeta, \ \zeta \in Z\}$ be arbitrary bounded in the $G(\psi)$ sense set of a functions:

$$\sup_{\zeta\in Z} ||f_\zeta||G(\psi) = \sup_{\zeta\in Z} \sup_{p\geq 2} |f_\zeta|_p/\psi(p) = C < \infty,$$

then

$$\sup_{\zeta\in Z} |f_\zeta|_p/\theta(p) \leq C\epsilon(p) \to 0, \ p \to \infty.$$

It follows from previous theorem that $\forall \zeta \in Z \ f_\zeta \in G^0(\theta)$ and that the family $\{f_\zeta, \ \zeta \in Z\}$ has uniform absolute continuous norm. Our assertion follows from lemma 13.3 in the book [6].

    B). Inverse, let $\Phi(\cdot) << N(\cdot)$. Let us introduce the measurable function $f : X \to R$ such that $\forall x \geq \exp(2)$

$$\exp\left(-2 [p\log\psi(p)]^* (\log x)\right) \leq T(|f|, x) \leq$$

$$\exp\left(- [p\log\psi(p)]^* (\log x)\right).$$

Then (see theorem 7)

$$f(\cdot) \in G(\psi), \quad C_{15}(\psi) \ \psi(p) \leq |f|_p \leq C_{14}(\psi) \ \psi(p), \quad p \geq 2.$$

Since $f \in G(\psi)$, $\Phi << N$, we deduce that $f \in G^0(\theta)$, and, following,

$$\lim_{p\to\infty} |f|_p/\theta(p) = 0.$$

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Therefore, $\lim_{p\to\infty} \psi(p)/\theta(p) = 0$.

Theorem 10. *Let now $X = R^d$ and $\psi \in \Psi$. We assert that the norms $|| \cdot ||L(N^{(\alpha)}, [\psi])$ and $|| \cdot ||G(\alpha, \psi)$, $\alpha \geq 1$ are equivalent.*

Proof. 1. Let $\forall p \geq \alpha \Rightarrow |f|_p \leq \psi(p)$, $f \neq 0$. From Tchebychev's inequality follows that

$$\lim_{v\to\infty} T(|f|, v) = 0.$$

Let us consider for some sufficiently small value $\epsilon \in (0, \epsilon_0)$, $\epsilon_0 \in (0, 1)$ the following integral:

$$I_{\alpha,N}(f) = \int_X N^{(\alpha)}(\epsilon|f(x)|) \ dx = I_1 + I_2,$$

where

$$I_1 = \int_{\{x:|f(x)|\leq v\}} N^{(\alpha)}(\epsilon|f(x)|) \ dx, \ I_2 = \int_{\{x:|f(x)|>v\}} N^{(\alpha)}(\epsilon|f(x)|) \ dx.$$

Since for $z \geq v$

$$N^{(\alpha)}(z) \leq C_{19}(\alpha, N(\cdot)) \cdot N(z),$$

we have for the set $X(v) = \{x, |f(x)| > v\}$ and using the result of theorem 7 for the space with finite measure:

$$I_2 = \int_{X(v)} N^{(\alpha)}(\epsilon|f(x)|) \ dx \leq C_{20}(\alpha, N, \epsilon) \ ||f||L(N^{(\alpha)}, X(v)) \leq$$

$$C_{21}(\alpha, \epsilon, \psi) \sup_{p\geq\alpha} [||f||L_p(X(v))/\psi(p)] \leq C_{21} \sup_{p\geq\alpha} |f|_p/\psi(p) < \infty.$$

Further, since for $z \in (0, v) \Rightarrow$

$$N^{(\alpha)}(\epsilon z) \leq C_{22}(v, \alpha, \epsilon) \ |z|^\alpha,$$

we have:

$$I_1 \leq C_{22}(\cdot) \int_X |f(x)|^\alpha \ dx = < \infty.$$

Thus, $f \in L(N^{(\alpha)}[\psi])$, $||f||L(N^{(\alpha)}[\psi]) < \infty$.

2). We prove now the inverse inclusion. Let $f \in L\left(N^{(\alpha)}[\psi]\right)$ and

$$||f||L\left(N^{(\alpha)}[\psi]\right) = 1.$$

Hence for some $\epsilon > 0$

$$\int_X N^{(\alpha)}(\epsilon|f(x)|)dx < \infty.$$

It follows from the proof of theorem 7 and the consideration of two cases: $|z| \leq v$; $|z| > v$ the following elementary inequality: at $p \geq \alpha$ and for all $z > 0 \Rightarrow$

$$|z|^p \leq C_{23}(\alpha, \epsilon, N) \ N^{(\alpha)}(\epsilon|z|) \cdot \psi^p(p).$$

We obtain for all values p, $p \geq \alpha$:

$$\int_{R^d} |f(x)|^p \ dx \leq C_{24}^p(\alpha, \epsilon, \psi) \ \psi^p(p), \quad ||f||G(\alpha; \psi) < \infty.$$

7 Proofs of main results.

At first we consider the case Orlicz spaces, i.e. if the function f belongs to some exponential Orlicz space.

Proof of theorems 1,2. Let $X = [0, \ 2\pi]^d$ and $f \in L(N[\psi])$ for some $\psi \in \Psi$. Without loss of generality we can assume that $||f||L(N[\psi]) = 1$. From theorem 7 follows that

$$\forall p \geq 2 \ \Rightarrow \ |f|_p \leq C_{25}(\psi) \ \psi(p).$$

From the classical theorem of M. Riesz follows the inequality:

$$\sup_{M \geq 1} |s_M[f]|_p \leq K_1^d \ p^d \ \psi(p), \ \ K_1 = 2\pi.$$

It follows again from theorem 7 that

$$\sup_{M \geq 1} ||s_M[f] - f||L(N_d[\psi]) \leq K_1^d + 1 < \infty.$$

For example, if $N(u) = N_m(u) = \exp(|u|^m) - 1$ for some $m = \text{const} \geq 1$, then

$$\sup_{M \geq 1} ||s_M[f] - f||L(N_{m/(dm+1)}) \leq C_{26}(d, m) \ ||f||L(N_m). \tag{7.1}$$

The "continual" analog of M.Riesz's inequality, namely, the case $X = R$, $L(N) = L_p(R)$, $p \geq 2$:

$$\sup_{M \geq 2} |S_M[f]|_p \leq K_2^d \ p^d \ |f|_p, \ \ K_2 = 1$$

is proved, for example, in [46], p.187 - 188.

This fact permit us to prove also theorem 2.

Lemma 1. *We assert that the "constant" $m/(dm + 1)$ in the estimation (3.5) is exact. In detail, for all $m \geq 1$ there exists $g = g_m(\cdot) \in L(N_m)$ such that $\forall \Delta \in (0, 1/2)$*

$$\sup_{M \geq 1} ||s_M[g]||L\left(N_{(m-\Delta)/(dm+1)}\right) = \infty.$$

Proof of lemma 1. It is enough to prove that

$$\exists g \in L(N_m), \ \ ||H[g]||L(N_{(m-\Delta)/(dm+1)}) = \infty,$$

where $H[g]$ denotes the Hilbert transform on the $[0, \ 2\pi]^d$, see [7], p. 193 - 197. Also it is enough to consider the case $d = 1$.

Let us introduce the function

$$g(x) = g_m(x) = |\log(x/(2\pi))|^{1/m}.$$

Since for $u > 0$

$$\text{mes}\{x : g_m(x) > u\} = \exp(-u^m),$$

we conclude $g_m(\cdot) \in L(N_m) \setminus L^0(N_m)$ (theorem 7). Further, it is very simple to verify using the formula for Hilbert transform that

$$C_{28}(m) \left(|\log(x/(2\pi))|^{(m+1)/m} + 1\right) \leq |H[g_m](x)| \leq$$

$$C_{29}(m) \left(|\log(x/(2\pi))|^{(m+1)/m} + 1 \right).$$

Hence $\forall u \geq 2$

$$\exp\left(-C_{29}(m)u^{m/(m+1)}\right) \leq \text{mes}\{x, |H[g_m](x) > u\} \leq$$

$$\exp\left(-C_{30}(m) \ u^{m/(m+1)}\right).$$

It follows again from theorem 7 that

$$H[g_m] \in L(N_{m/(m+1)}) \setminus L(N_{m/(m+1)}).$$

Thus $\forall \Delta \in (0,) \Rightarrow H[g_m] \notin L(N_{(m-\Delta)/(m+1)})$.

Proof of theorem 3. Let us consider the following function:

$$z(x) = z_L(x) = \sum_{n=8}^{\infty} n^{-1} \ L(n) \ \sin(nx). \tag{7.2}$$

It is known from the properties of slowly varying functions ([14], p. 98 - 101) that the series (7.2) converge a.e. and at $x \in (0, \ 2\pi]$

$$C_0 L(1/x) \leq z(x) \leq CL(1/x).$$

Therefore, at $u \in [\exp(2), \infty)$

$$L^{-1}(Cu) \leq T(|z|, u) \leq L^{-1}(C_0 u).$$

It follows from theorem 7 and (5.2) that

$$z(\cdot) \in L(N) \setminus L^0(N), \ N(u) = L^{-1}(u), \ u \geq \exp(2).$$

From theorem 8 follows that

$$0 < C_0 \leq |z|_p/\psi(p) \leq C < \infty, \ p \geq 2, \ \psi(p) = \exp(W^*(p)/p). \tag{7.3}$$

Note as a consequence that the series (7.2) *does not converge in the $L(N)$ norm,* as long as the system of functions $\{\sin(nx)\}$ is bounded and hence in the case when the series (7.2) *converge* in the $L(N)$ norm $\Rightarrow z(\cdot) \in L^0(N)$.

Let us suppose now that for some EOF $\Phi(\cdot)$ with correspondence function $\theta(p)$ (7.2) convergence in the $L(\Phi)$ norm. Assume converse to the assertion of theorem 3, or equally that

$$\overline{\lim}_{p\to\infty}\theta(p)/\psi(p) > 0. \tag{7.4}$$

Since the system of functions $\{\sin(nx)\}$ is bounded, $z(\cdot) \in L^0(\Phi)$. By virtue of theorem 8 we conclude that

$$\lim_{p\to\infty} |z|_p/\theta(p) = 0.$$

Thus, we obtain from (7.3)

$$\lim_{p\to\infty} \psi(p)/\theta(p) = 0,$$

in contradiction with (7.4). The cases $X = [0, 2\pi]^d$, $X = R^d$ are considered as well as the case $X = [0, 2\pi]$.

Now we consider the case when $f \in G(a, b, \alpha, \beta)$.

Proof of theorem 4. Let $f \in G\psi$, $\psi \in G\Psi(1, 2)$, $||f||G\psi = 1$; then

$$|f|_q \leq \psi(q), \ q \in (1, 2).$$

We denote $p = q/(q - 1)$, then $p \in [2, \infty)$. We will use the classical result of Hardy - Littlewood, Hausdorff - Young [33], p.193; [47], p. 93:

$$|F[f]|_p \leq C(d) \ |f|_q \leq C(d)\psi(q) = C(d)\psi(p/(p-1)) = C(d)\zeta(p),$$

or equally

$$||F[f]||G\zeta \leq C(d) = C(d) \ ||f||G\psi.$$

In order to prove the exactness of theorem 4, we consider the following example. Let $d = 1$, $X = R^1$,

$$f_0(x) = f(x) = |x|^{-1} \ I(|x| \geq 1).$$

We deduce for the values $q \in (1, 2)$ and following $p \in (2, \infty)$; $q \to 1+0 \ \Rightarrow p \to \infty$:

$$|f|_q^q = 2 \int_1^\infty x^{-q} \ dx = 2 \ (q - 1)^{-1},$$

$$q \to 1 + 0 \Rightarrow |f|_q \sim 2 \ (q - 1)^{-1};$$

$$F[f](t) := F(t) = 2 \int_1^\infty x^{-1} \ \cos(tx) \ dx.$$

Note that as $t \to 0+$

$$F(t) = 2 \int_t^\infty y^{-1} \ \cos(y) \ dy \sim 2 \int_t^1 y^{-1} \ \cos(y) \ dy \sim$$

$$2 \int_t^1 dy/y = 2|\log t|.$$

Therefore, as $p \to \infty$

$$|F|_p^p \sim 2^p \int_0^1 |\log t|^p \ dt = 2^p \ \Gamma(p + 1),$$

$$|F|_p \sim= \Gamma^{1/p}(p + 1) \sim C_2 \ p, C_2 = e^{-1}.$$

We conclude after dividing:

$$\overline{\lim}_{p \to \infty} \frac{F[f_0]_p}{|f_0|_q} \geq C_2/2 > 0,$$

The second proposition of this theorem follows from theorem 10.

Proof of theorem 4a may be ground alike the proof of of theorem 4 with the analogous counterexample; the Hardy-Young inequality for the Fourier series has a view

$$|F[c]|_p \leq C_1(d) \ |c|_{q,d}, \ q \in (1,2), \ p = q/(q-1).$$

Proof of theorem 5A. Upper estimation. We will use the classical result of W.Paley and F.Riesz ([47], p.120).

Let $\{\phi_k(x), \ k = 1, 2, \ldots\}$ be some orthonormal bounded sequence of functions. Then $p \geq 2 \ \Rightarrow$

$$|f|_p \leq K_3 \cdot p \ \cdot \left(1 + \sup_{k,x} |\phi_k(x)|\right) \cdot \left(\sum_k |c(k)|^p \left(|k|^{p-2} + 1\right)\right)^{1/p}, \qquad (7.5)$$

where K_3 is an absolute constant, $f(x) = \sum_k c(k) \ \phi_k(x)$.

Let $c(\cdot) \in g(\psi, \nu)$ and $||c|| g(\psi, \nu) = 1$. By definition of the $|| \cdot || g(\psi, \nu)$ norm

$$\sum_{k=1}^{\infty} |c(k)|^p \left(k^{p-2} + 1\right) \leq \ ||c||^p g(\psi, \nu) \cdot \psi^p(p).$$

Therefore

$$|f|_p \leq K_3 \ C_{31} \ p \cdot \psi(p),$$

and by virtue of theorem 7 $f(\cdot) \in L(N_1[\psi])$.

Proof of theorem A. Exactness.

Let us consider the following example:

$$g(x) = \sum_{n=2}^{\infty} n^{-1} \ \log^m n \ \cos(nx), \ x \in [-\pi, \pi].$$

$m = \text{const} > 0$. Here

$$c(n) = n^{-1} \ \log^m n,$$

and following as $p \to \infty$

$$|c|_{p,\nu}^p = \sum_{n=2}^{\infty} h^{-2} \ \log^{pm} n \ \sim \int_1^{\infty} x^{-2} \ \log^{pm} x \ dx = \Gamma(pm + 1);$$

$$|c|_{p,\nu} \sim [\Gamma(pm + 1)]^{1/p} \sim m^m e^{-m} \ p^m.$$

We know that

$$|g(x)| \sim C(m) \ |\log |x| \ |^{m+1}, x \to 0; \ |g|_p \sim C_1(m) \ p^{m+1}.$$

Substituting into the expression for the value $\overline{V}$, we get to the assertion of our theorem.

Proof of Theorem 5 B. Here we use the "discrete" inequality of Hausdorff-Young, Hardy - Littlewood (see [7], p.101; [12], [20], chapter 5, [42], chapter 4, sections 1,2:

$$|f|_p \le K_4 \, |c|_q, \ p \ge 2, \ q = p/(p-1), \quad K_4 = 2\pi.$$

If $||c||g(\alpha) = 1$, then

$$|c|_q \le (q-1)^\alpha, |f|_p \le K_4 \, p^\alpha, \ p \ge 2.$$

Again from theorem 7 follows that $f \in L\left(N_{1/\alpha}\right)$.

Proof of Theorem 5 C is at the same as the proof of theorem 5. We use the following classical inequality:

$$|f|_{p/(p-1)} \le (1+M) \, |c|_p,$$

see [16], chapter 6, section 3.

Proof of theorem 6. The analog of inequality (7.5) in the case

$$F[f](t) = \int_R \exp(itx) f(x) dx, \ d = 1,$$

namely:

$$|F[f]|_p \le K_5 \, p \, |f|_p(\nu),$$

when $f(\cdot) \in L_p(\nu) \subset G(\nu, \alpha, \psi)$, see, for example, in [23], p. 108. Hence, for all $p \ge \alpha$

$$||F[f]||L\left(N_d^{(\alpha)}[\psi]\right) \le K_5 \, ||f||G(\alpha; \psi, \nu).$$

(The generalization on the case $d \ge 2$ is evident).

Note that the moment estimations for the wavelet transforms and Haar series are described for example in the books [7], p.21, [33], p.297 etc.

It is easy to generalize our results on the cases Haar's or wavelet series and transforms.

In detail, it is true in this cases the moment estimation for the partial sums (wavelet's or Haar's)

$$|P_M[f]|_p \le K_6 \, |f|_p, \ X = [0,1], \ p \ge 1,$$

where $K_6 = 13$ for Haar series on the interval $X = [0,1]$ and $K_6 = 1$ for the classical wavelet series, in both the cases $X = [0,1]$ and $X = R$. Hence $\forall \psi \in \Psi, \ f(\cdot) \in L(N[\psi])$

$$\sup_{M \ge 1} ||P_M[f] - f||L(N[\psi]) \le (K_6 + 1) \, ||f||L(N[\psi]). \tag{7.6}$$

But (4.6) does not mean in general case the convergence

$$\lim_{M \to \infty} ||P_M[f] - f||L(N[\psi]) = 0, \tag{7.7}$$

as long as if (7.7) is true, then $f(\cdot) \in L^0(N[\psi])$ and conversely if $f \in L^0(N[\psi])$, then (7.7) holds.

For the different generalizations of wavelet series the estimation (7.6) with constants K_6 not depending on $p, \ p \ge 2$ see, for example, in the books [7], [33], [47] etc.

8   Concluding remarks. Maximal operators.

We consider in this section the so-called *maximal Fourier operators* and investigate their boundedness in some Grand Lebesgue spaces.

Let us define the following maximal operators in the one-dimensional case $d = 1$:

$$s^*[f](x) = \sup_{M \geq 1} |s_M[f](x)|, \tag{8.1},$$

$$F^*[f](x) = \sup_{a > 0} \left| \int_{-a}^{a} f(t) \exp(itx)\ dt \right|, \tag{8.2},$$

$$R^*[f](x) = \sup_{a > 0} \left| \int_{-\infty}^{\infty} f(t) \frac{\sin(a(x - t))}{x - t}\ dt \right|. \tag{8.3}$$

Theorem M1. Let $f \in G\psi, \operatorname{supp} \psi = (1, \infty)$. We define

$$\psi_{\lambda,\mu}(p) = \frac{p^{\lambda}}{(p - 1)^{\mu}}\ \psi(p).$$

We assert:

$$||s^*[f]||G\psi_{4,3} \leq K_{4,3}\ ||f||G\psi, \tag{8.4},$$

$$||R^*[f]||G\psi_{4,2} \leq K_{4,2}\ ||f||G\psi. \tag{8.5}$$

Theorem M2. Let $f \in G\psi, \operatorname{supp} \psi = (1, 2)$. We define as before

$$\zeta(q) = q^2\ \psi(q/(q - 1)),\ q \in (2, \infty).$$

Assertion:

$$||F^*[f]||G\zeta \leq K_5\ ||f||G\psi. \tag{8.6}$$

Proof is at the same as before. It used the following maximal L_p Fourier estimations:

$$|s^*[f]|_p \leq K_{4,3}\ |f|_p\ \frac{p^4}{(p - 1)^3},\ p \in (1, \infty);$$

$$|R^*[f]|_p \leq C\ K_{4,2}\ |f|_p\ \frac{p^4}{(p - 1)^2},\ p \in (1, \infty);$$

$$|F^*[f]|_q \leq K_5\ |f|_p\ (p - 1)^{-2},\ p \in (1, 2),\ q = p/(p - 1);$$

see, e.g., the classical monograph of Reyna [37], p. 144-152; or [1].

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